Mathematics for Engineers II

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Seminar
Differential equations

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Numerical differentiation with Matlab

Example

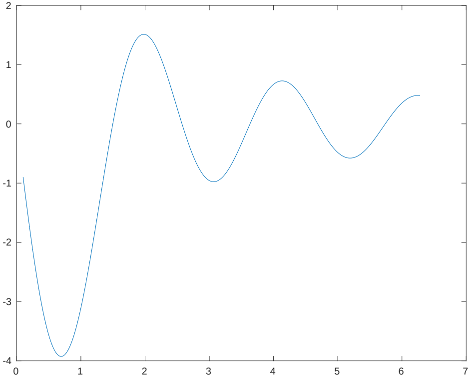
Assume that we have the value of $f(x) = \frac{sin(3x)}{x}$ on the interval $[0.1, 2\pi]$ with step size $h = 0.001$. Calculate the derivative numerically, and plot the approximative and the real derivative on the same figure.

Solution: Use the `diff` function.

If `y` is an n-tuple vector, then `diff(y)` is an $(n-1)$-tuple vector, it contains the difference of `y`'s consecutive elements:
[y(2)-y(1), y(3)-y(2), ..., y(n)-y(n-1)]

```
>> h=0.001;
>> x=0.1:h:2*pi;
>> y=sin(3*x)./x;
>> d1=diff(y)./diff(x);
>> figure; plot(x(1:end-1),d1)
```

```
h=0.001;
x=0.1:h:2*pi;
y=sin(3*x)./x;
d1=diff(y)./diff(x);
figure; plot(x(1:end-1),d1)
```



The derivative of f is:

$$f'(x) = \frac{3\cos(3x)}{x} - \frac{\sin(3x)}{x^2}$$

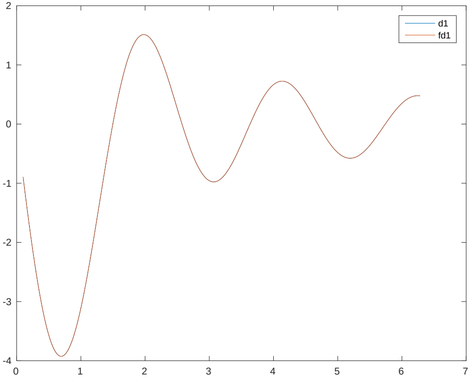
Calculate its values at the coordinates of `x`, and plot them on the previous figure:

```
>> fd1=3*cos(3*x)./x-sin(3*x)./x.^2;
>> hold on; plot(x,fd1); hold off
```

One can calculate the largest difference between the coordinates of `d1` and `fd1`. (Be careful, `d1` has less coordinates than `fd1`.)

```
>> max(abs(fd1(1:end-1)-d1))
```

```
h=0.001;
x=0.1:h:2*pi;
y=sin(3*x)./x;
d1=diff(y)./diff(x);
figure; plot(x(1:end-1),d1)
fd1=fd1=3*cos(3*x)./x-sin(3*x)./x.^2;
hold on; plot(x,fd1); hold off
```



Exercises

(1) Differentiate $f(x) = \frac{sin(3x)}{x}$ numerically changing the value of h e.g. $h = 0.01$, $h = 0.005$, $h = 0.001$. Calculate the largest difference between the numerical and the exact value.

(2) Let's calculate the values of $f(x) = sin(\frac{100}{x})$ on the interval $[0.5, 2\pi]$ with step size $h = 0.001$! Calculate the numerical derivative! Plot the numerical and the exact derivative on the same figure! Explain the difference between this case and the case of $\frac{sin(3x)}{x}$.

(3) Solve the previous exercise with step size $h = 0.001$ using the formula below for the numerical derivative:

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

What is the difference?

Approximation of the second derivative with Matlab

Apply the diff function twice.

```
h=0.001;
x=0.1:h:2*pi;
y=sin(3*x)./x;
d1=diff(y)./diff(x);
d2=diff(d1)./diff(x(1:end-1);
figure; plot(x(1:end-2),d2)
```

Exercises for differential equations

1. Example

A flask contains 100 liters of water and to it is being added a salt solution that contains 0.1 kilograms of salt per liter. This salt solution is being poured in at the rate of 5 liters per minute. The solution is being thoroughly mixed and drained of and the mixture is drained off at the same rate so that the flask contains 100 liters at all times. How much salt is in the flask after an hour?

Solution:

- $y(t)$: it denotes the amount of salt at time t.
- $\Delta t := y(t + \Delta t) - y(t)$: it denotes the changing rate of salt.
- At time instant t there is $\frac{5}{100}y(t)$ kilograms salt in every 5 liters solution.
- If Δt is small enough, then $\frac{5}{100}y(t)\Delta t$ amount of salt is withdrawn from the flask in an interval Δt.

Exercises for differential equations

The equation is

$$y(t+\Delta t) - y(t) = -\frac{1}{20} y(t)\Delta t,$$

that is

$$\frac{y(t+\Delta t) - y(t)}{\Delta t} = -\frac{1}{20} y(t).$$

If $\Delta t \to 0$, then

$$y'(t) = -\frac{1}{20} y(t).$$

Solution of the ODE is

$$y(t) = C \cdot e^{-\frac{1}{20} t},$$

where the value of C comes from the initial condition $y(0) = 10$:

$$y(0) = C \cdot e^{-\frac{1}{20} \cdot 0} = C = 10,$$

so

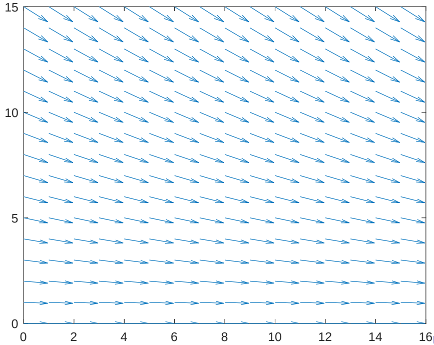
$$y(t) = 10 \cdot e^{-\frac{1}{20} t}, \text{ és } y(60) = 10 \cdot e^{-3} \approx 0.5$$

Direction field

If the solution $y(t)$ of the previous equation

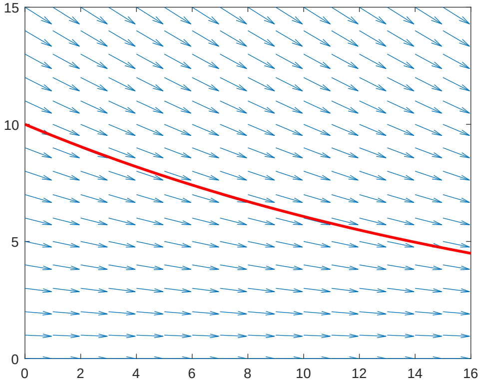
$$y'(t) = -\frac{1}{20}y(t)$$

goes trough a point of the plane (t_0, y_0), then its slope is $-\frac{1}{20}y_0$.
Create a uniformly spaced grid of points in a domain of the plane, and add a small line segment or arrow to every grid with slope $f(x, y)$:



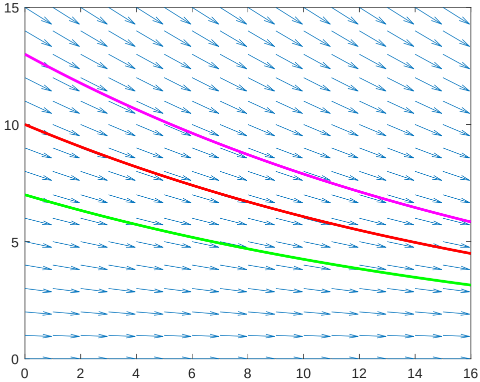
Direction field

Draw the solution belonging to the initial data $y(0) = 10$ in the directional field:



Direction field

If the initial data is $y(0) = 7$, or it is $y(0) = 13$:



Direction field Matlab

Usage of command `quiver`:

- `[T,Y]` contains the coordinates of the grid (starting points of the arrows),
- `[dT,dY]` contains the ending points of the arrows,

then the command

```
>> quiver(T,Y,dT,dY)
```

plot the Direction field.

In general, the directional field of $y'(t) = f(t, y(t))$ is

```
>> [T Y] = meshgrid(minT:stepsize:maxT, minY:stepsize:maxY);
>> dY = f(T,Y);
>> dT = ones(size(dY));
>> quiver(T,Y,dT,dY);
```

Example

Plot the directional field of $y'(t) = -\frac{1}{20}y(t)$ on the domain $[0, 15] \times [0, 15]$.

- Make a grid with step size one:

```
>> t=0:15; y=0:15;
>> [T,Y]=meshgrid(t,y);
```

Then both T and Y will be a 16×16 matrices:

$$T = \begin{bmatrix} 0 & 1 & \dots & 14 & 15 \\ 0 & 1 & \dots & 14 & 15 \\ \vdots & & & & \\ 0 & 1 & \dots & 14 & 15 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & & & & \\ 15 & 15 & \dots & 15 & 15 \end{bmatrix}$$

- Generate a line element for the all coordinates of `[T,Y]`:

```
>> dY=-Y/20;
>> dT=ones(size(dY));
```

- Plot the Direction field:

```
>> quiver(T,Y,dT,dY);
```

Summary:

```
>> t=0:15; y=0:15;
>> [T,Y]=meshgrid(t,y);
>> dY=-Y/20;
>> dT=ones(size(dY));
>> quiver(T,Y,dT,dY);
```

Exercise

(a) Plot the directional field of

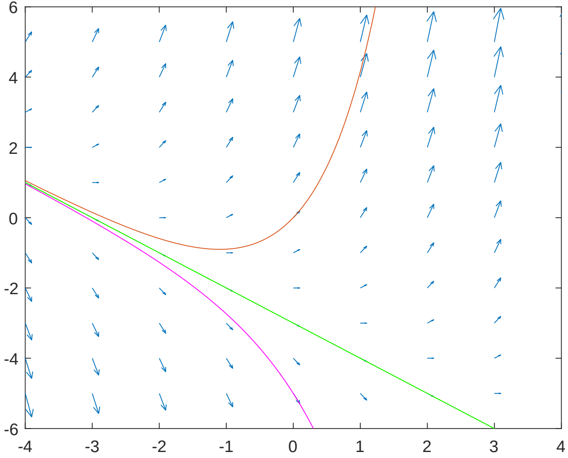
$$y' = x + y + 2$$

on the domain $[-4, 4] \times [-6, 6]$.

(b) Check the function $y(x) = -x - 3 + k \cdot e^x$ is a solution of the previous equation (k is an arbitrary constant).

(c) Plot into the directional field the solutions belonging to the following initial data:

- $y(0) = 0$,
- $y(0) = -3$,
- $y(0) = -5$.


6
4
2
0
-2
-4
-6
-4
-3
-2
-1
0
1
2
3
4

Examples

2. Example

A boot decelerates because of the counter-move of the river. Its starting velocity is 1.5 m/s, after 4 s it is 1 m/s. How soon does it decelerate 1 cm/s, if the counter-move is proportional to the velocity?

Solution:

v(t): velocity of the boot at t in m/s

The ODE:

$$v'(t) = k \cdot v(t)$$

Initial conditions:

$$v(0) = 1.5 \quad \text{és} \quad v(4) = 1.$$

Examples

Solution of the ODE:

$$v(t) = C \cdot e^{kt}.$$

Using the initial conditions:

$$v(0) = C \cdot e^{k \cdot 0} = C = 1.5,$$

$$v(4) = 1.5 \cdot e^{k \cdot 4} = 1 \implies k = \frac{1}{4} \ln \frac{1}{1.5} = -0.1014$$

So the solution is:

$$v(t) = 1.5 \cdot e^{-0.1014t}$$

How soon will be the velocity 1 cm/s=0.01 m/s?

$$0.01 = 1.5 \cdot e^{-0.1014t} \implies t = -\frac{1}{0.1014} \ln \frac{0.01}{1.5} \cdot = 49.415$$

Symbolic solution with Matlab

Example

Let's solve the previous example with the Matlab function `dsolve`!

One can solve with the command `dsolve(eqn)` the equation `eqn` if it is given in a symbolic way.
Our equation is

$$v'(t) = k \cdot v(t).$$

Let's define k and $v(t)$ symbolically:

```
>> syms k v(t)
```

Give the equation in a symbolic way:

```
>> eqn= diff(v,t)==k*v
```

Symbolic solution with Matlab

Solution with the function `dsolve`:

```
>> dsolve(eqn)
ans=
   C1*exp(k*t)
```

This is the general solution. We can solve an initial value problem with the initial condition $v(0) = 1.5$ in the following way:

```
>> syms k v(t);
>> eqn= diff(v,t)==k*v;
>> cond1= v(0)==1.5;
>> dsolve(eqn,cond1)
ans=
   (3*exp(k*t))/2
```

Exercises

(1) A body cools in 10 minutes from 100 Celsius to 60. The surroundings are at a temperature of 20 Celsius. When will the body cool to 25 if the rate of change of temperature is proportional to the difference of temperature between a cooling body and its surroundings?

(2) Find a curve such that the point of intersection of an arbitrary tangent with the x axis has an abscissa half as great as the abscissa of the point of tangency.

(3) A flask contains 10 liters of water and being added a salt solution that contains 0.3 The solution is being thoroughly mixed and drained of and the mixture is drained off at the same rate so that the flask contains 10 liters at all times. How much salt is in the flask after 5 minutes?

(3) A large chamber contains 200 cubic meters of gas, 0.15% of which is carbon dioxide ($C0_2$). A ventilator exchanges 20 cubic meters per minute of this gas with new gas containing only 0.04% CO_2. How long will it be before the concentration of CO_2 is reduced to third its original value?

Classification of differential equations

Exercises

Choose the PDEs and ODEs from the equations below!

(a) $x'x''x''' = \frac{\partial y}{\partial t_1}$,

(b) $x'x''x''' = \cos(t)$

(c) $\frac{\partial y}{\partial t} - \triangle y = 0$

(d) $a\ddot{x} = \dot{x}$

(e) $T' = -kT$

(f) $\partial_1 z + \partial_2 z = -\partial_1^2 z$

(g) $x'' - \cos(t)x' + x = e^x$

Classification of differential equations

Exercises

Which one is linear and which one is nonlinear from the following equations. Which term does cause nonlinearity?

(a) $y' = x + y + 2$

(b) $x' - xt^2 = 3t^3$

(c) $y' - y^2 x + x = 0$

(d) $x' = x\cos(t) + t^2$

(e) $x' - te^x + sin(t) = 0$

(f) $u'(1 + t) = u\cos(t) - u - e^t$

Easily integrable equations

Exercises

Adja meg a következő egyenletek általános megoldását!

(a) $x' = sin(t)$

(b) $y' = \frac{1}{1+x}$

(c) $y' = \sqrt{x+2}$

(d) $u' = t(t+1)$

(e) $x' = e^{t-3}$

(f) $y' = (2t+1)^3$

(g) $x' = cos(t) + t^2 - e^t$

Separable ODEs

Exercises

Solve the ODEs and initial value problems below!

(a) $x' = t \cdot x$

(b) $y' = e^{y+2}$

(c) $u' = u(u+1)$

(d) $y' = 1 - y$

(e) $y' = \frac{x^2}{y+2}$

(f) $(1+x)yy' = 1$

(g) $3u' + \cos(x)u^2 = 0$

(h) $t(t-1)x' + x(x-1) = 0$

Exercises

Solve the previous exercises with Matlab symbolically!

Linear inhomogeneous equations

Example

Solve the differential equation $x(y' - y) = e^x$!

Solution: Rearranging the equation we get the inhomogeneous, linear ODE:

$$y' = y + \frac{e^x}{x}.$$

- Solve at first the homogeneous part $y' = y$!
  Its solution is $y = C \cdot e^x$, where C is a constant.
- Variation of constants: let's consider c as a function of x:

$$y = C(x) \cdot e^x$$

Linear, inhomogeneous equations

Substitute this into the original equation:

$$y' = C'(x) \cdot e^x + C(x) \cdot e^x = C(x) \cdot e^x + \frac{e^x}{x}$$

From this we have

$$C'(x) \cdot e^x = \frac{e^x}{x} \implies C'(x) = \frac{1}{x},$$

that is

$$C(x) = \ln|x| + K,$$

where K is a constant.

A megoldás tehát:

$$y = (\ln|x| + K)e^x$$

Linear, inhomogeneous equations

Exercises

Solve the following equations!

(1) $y' = x + y + 2$

(2) $xy' - 2y = 2x^4$

(3) $x^2y' + xy + 1 = 0$

(4) $y' = 2x(x^2 + y)$

Exercises

Solve the previous equations with Matlab!

Bernoulli's equation

Example

Let's solve the equation $y' + 2y = y^2 e^x$.

Solution: It has a form $y' + g(x)y + h(x)y^{\alpha} = 0$, so this is a Bernoulli type equation ($g(x) \equiv 2$, $h(x) = -e^x$, $\alpha = 2$), $y = 0$ is the trivial solution.

Multiply the equation by $(1-\alpha)y^{-\alpha} = -y^{-2}$:

$$-\frac{y'}{y^2} - \frac{2}{y} = -e^x$$

Let $z = y^{1-\alpha} = y^{-1}$. Then $z' = -y^{-2}y'$ and

$$z' - 2z = -e^x$$

This is an inhomogeneous, linear equation.

- Solution of the homogeneous part $z' - 2z = 0$ is $z = C \cdot e^{2x}$
- Variation of constant:

$$z' = C' \cdot e^{2x} + 2C \cdot e^{2x}$$

from this we get

$$\underbrace{C' \cdot e^{2x} + 2C \cdot e^{2x}}_{z'} - 2\underbrace{C \cdot e^{2x}}_{z} = -e^{x} \implies C' \cdot e^{2x} = -e^{x}$$

so

$$C' = -e^{-x} \implies C = e^{-x} + K$$

Solution of the inhomogeneous equation is

$$z = e^{x} + Ke^{2x}.$$

Solution of the original equation is

$$y = \frac{1}{e^{x} + Ke^{2x}} \quad \text{és} \quad y = 0.$$

Bernoulli's equation

Exercises

Solve the following equations!

(a) $xy^2y' = x^2 + y^3$

(b) $y' = y^4 \cos(x) + y \tan(x)$

Exercise

Solve the previous equations with Matlab.

Riccati equation

Exercise

Solve the following Riccati equations with the help of the given particular solutions!

(a) $y'(x) + 2y(x)e^x - y^2(x) = e^{2x} + e^x, \quad y_p(x) = e^x$

(b) $y'(x) - \frac{y(x)}{x} = y^2(x) + \frac{1}{x^2}, \quad y(x)_p = \frac{c}{x}$

Exercise

Solve the previous equations with Matlab.

Second order ODEs with constant coefficients

Exercise

Solve the homogeneous equations "by hand" and with Matlab! Solve the inhomogeneous equations with Matlab and check with derivation that the solutions fulfil the equations.

a. $y''(x) - y'(x) - 2y(x) = 0,$
b. $y''(x) - 4y'(x) + 5y(x) = 0,$
c. $y''(x) + 6y'(x) + 9y(x) = 0, \quad y(0) = 1,\ y'(0) = 1,$
d. $y''(x) + 3y'(x) + 2y(x) = e^{3x},$
e. $y''(x) - 2y'(x) - 3y(x) = e^{4x},$
f. $y''(x) - y'(x) = 2e^{x} - x^2,$
g. $y''(x) - 3y'(x) + 2y(x) = \sin x,$
h. $y''(x) + 2y'(x) - 3y(x) = 0,$
i. $y''(x) + y'(x) - 6y(x) = 5e^{2x}, \quad y(0) = 0,\ y'(0) = 1.$

Higher order equations

Exercise

Give the equivalent system of ODEs to the following higher order equations!

(a) $y'''(x) - y'(x) = 0,\ y(0) = 3,\ y'(0) = -1,\ y''(0) = 1,$
(b) $y^{(4)}(x) - y(x) = 0,$
(c) $2xy''(x) + y'(x) = 0,$
(d) $2y(x)y'(x) = y''(x),$
(e) $y''(x) = \frac{(y'(x))^2}{y(x)},$
(f) $y'''(x) - 8y(x) = 0,$
(g) $y^{(4)}(x) + 2y''(x) + y(x) = 0,$
(h) $y'''(x) - 3y''(x) + 3y'(x) - y(x) = 0,$
(i) $xy'''(x) - y''(x) - xy'(x) + y(x) = 0.$